

# Four types of phase transitions in interacting meson (boson) matter at high temperatures

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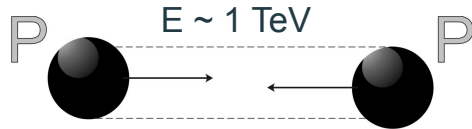
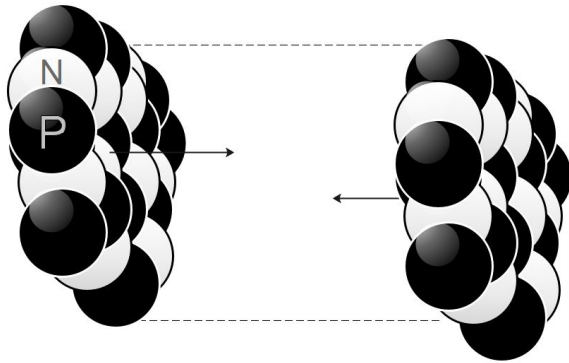
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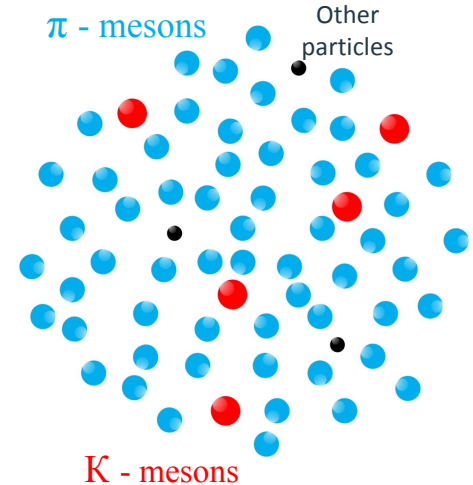
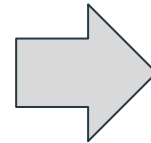
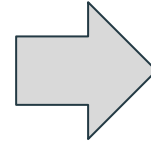
# Outline

- Introduction
- Creation of the Bose-Einstein condensate: phase transitions of the 2nd order
- Selfinteracting scalar field
- Thermodynamic mean-field model for a particle-antiparticle system
- "Weak" and "Strong" attraction
- Conserved isospin density: Four types of phase transitions
- Concluding remarks

# High energy collisions



Task: Study of the conditions for the occurrence of Bose condensate in the pion-antipion system with interaction

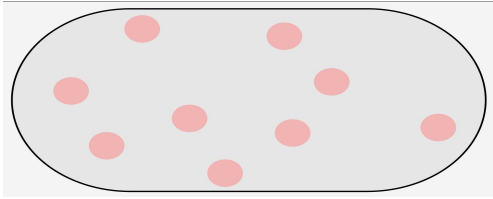


$N \sim 10^2 - 10^4$

## Motivation

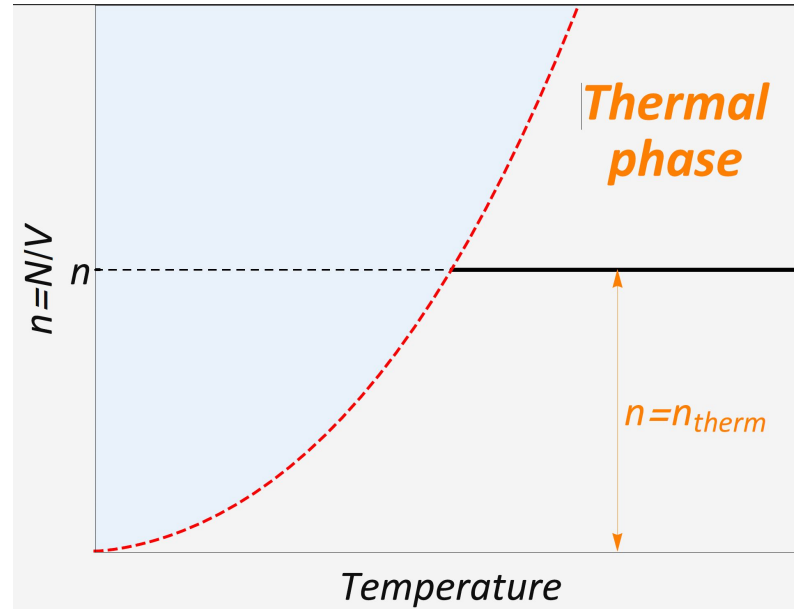
Knowledge of the phase structure of meson systems in the regime of finite temperatures and isospin densities is crucial for understanding a wide range of phenomena, from nucleus-nucleus collisions to neutron stars, as well as cosmology. This field is an important part of hot and dense hadronic matter research. At the same time, the study of meson systems has its own specificity due to the possibility of the Bose-Einstein condensation of bosonic particles.

# Introduction



$$n = \int \frac{d^3k}{(2\pi\hbar)^3} \frac{1}{\exp\left[\frac{k^2/2m - \mu}{T}\right] - 1}$$

$$f_{\text{BE}}(T, \mu) = \frac{1}{\exp\left[\frac{k^2/2m - \mu}{T}\right] - 1}$$



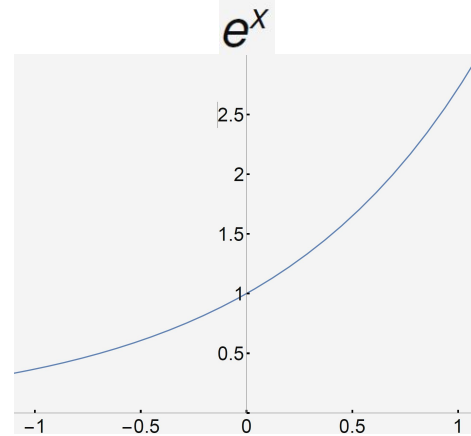
# Introduction (Limitation on chemical potential)

$$f_{\text{BE}}(x) = \frac{1}{e^x - 1}$$

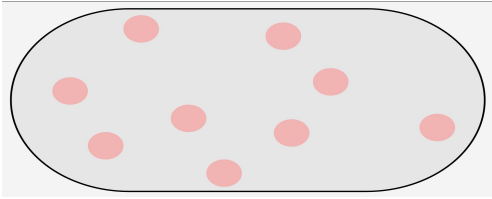
$$f_{\text{BE}}(E_j, \mu) = \frac{1}{\exp\left[\frac{E_j - \mu}{T}\right] - 1}$$

$$E_0 < E_1 < E_2 < \dots \rightarrow E_0 - \mu \geq 0$$

$$\Rightarrow \mu \leq E_0$$

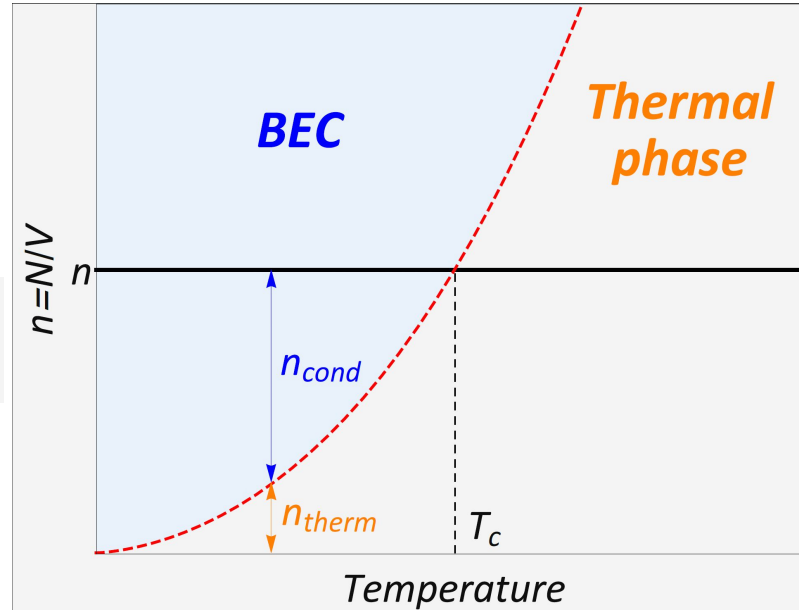


# Introduction



$$\frac{N}{V} = n_{\text{cond}} + \int \frac{d^3k}{(2\pi\hbar)^3} \frac{1}{\exp\left[\frac{k^2/2m - \mu}{T}\right] - 1} \Big|_{\mu=0}$$

$$f_{\text{BE}}(T, \mu) = \frac{1}{\exp\left[\frac{k^2/2m - \mu}{T}\right] - 1}$$



$$k_B T \rightarrow T$$
$$1 \text{ eV} = 1.16 \cdot 10^4 \text{ K}$$

# Original papers of Bose and Einstein

Bose S.N., Plancks Gesetz und Lichtquantenhypothese,

Zeitschrift für Physik, 26, 178-181, (1924).

## Zusammenfassung

Der Phasenraum eines Lichtquants in bezug auf ein gegebenes Volumen wird in „Zellen“ von der Größe  $h^3$  aufgeteilt. Die Zahl der möglichen Verteilungen der Lichtquanten einer makroskopisch definierten Strahlung unter diese Zellen liefert die Entropie und damit alle thermodynamischen Eigenschaften der Strahlung.

### Summary

The phase space of a light quantum in relation to a given volume is divided into “cells” of size  $h^3$ . The number of possible distributions of the light quanta of a macroscopically defined radiation among these cells provides the entropy and thus all thermodynamic properties of the radiation.



## Plancks Gesetz und Lichtquantenhypothese.

Von **Bose** (Dacca-University, Indien).

(Eingegangen am 2. Juli 1924.)

Der Phasenraum eines Lichtquants in bezug auf ein gegebenes Volumen wird in „Zellen“ von der Größe  $h^3$  aufgeteilt. Die Zahl der möglichen Verteilungen der Lichtquanten einer makroskopisch definierten Strahlung unter diese Zellen liefert die Entropie und damit alle thermodynamischen Eigenschaften der Strahlung.

Plancks Formel für die Verteilung der Energie in der Strahlung des schwarzen Körpers bildet den Ausgangspunkt für die Quantentheorie, welche in den letzten 20 Jahren entwickelt worden ist und in allen Gebieten der Physik reiche Früchte getragen hat. Seit der Publikation im Jahre 1901 sind viele Arten der Ableitung dieses Gesetzes vorgeschlagen worden. Es ist anerkannt, daß die fundamentalen Voraussetzungen der Quantentheorie unvereinbar sind mit den Gesetzen der klassischen Elektrodynamik. Alle bisherigen Ableitungen machen Gebrauch von der Relation

$$q_\nu d\nu = \frac{8\pi\nu^3 d\nu}{c^3} E,$$

d. h. von der Relation zwischen der Strahlungsdichte und der mittleren Energie eines Oszillators, und sie machen Annahmen über die Zahl der Freiheitsgrade des Äthers, wie sie in obige Gleichung eingeht (erster Faktor der rechten Seite). Dieser Faktor konnte jedoch nur aus der klassischen Theorie hergeleitet werden. Dies ist der unbefriedigende Punkt in allen Ableitungen, und es kann nicht wundernehmen, daß immer wieder Anstrengungen gemacht werden, eine Ableitung zu geben, die von diesem logischen Fehler frei ist.

Eine bemerkenswert elegante Ableitung ist von Einstein angegeben worden. Dieser hat den logischen Mangel aller bisherigen Ableitungen erkannt und versucht, die Formel unabhängig von der klassischen Theorie zu deduzieren. Von sehr einfachen Annahmen über den Energieaustausch zwischen Molekülen und Strahlungsfeld ausgehend, findet er die Relation

$$q_\nu = \frac{\alpha_{m,n}}{e^{\frac{h\nu}{kT}} - 1}$$



# Original papers of Bose and Einstein

## Quantentheorie des einatomigen idealen Gases ( Einstein , 1924 )

EINSTEIN: Quantentheorie des einatomigen idealen Gases

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### Quantentheorie des einatomigen idealen Gases.

VON A. EINSTEIN.

Eine von willkürlichen Ansätzen freie Quantentheorie des einatomigen idealen Gases existiert bis heute noch nicht. Diese Lücke soll im folgenden ausgefüllt werden auf Grund einer neuen, von Hrn. D. BOSE erdachten Betrachtungsweise, auf welche dieser Autor eine höchst beachtenswerte Ableitung der PLANCKSchen Strahlungsformel gegründet hat<sup>1</sup>.

Der im folgenden im Anschluß an BOSE einzuschlagende Weg läßt sich so charakterisieren. Der Phasenraum eines Elementargebildes (hier eines einatomigen Moleküls) in bezug auf ein gegebenes (dreidimensionales) Volumen wird in »Zellen« von der Ausdehnung  $h^3$  eingeteilt. Sind viele Elementargebilde vorhanden, so ist deren für die Thermodynamik in Betracht kommende (mikroskopische) Verteilung durch die Art und Weise charakterisiert, wie die Elementargebilde über diese Zellen verteilt sind. Die »Wahrscheinlichkeit« eines makroskopisch definierten Zustandes (im PLANCKSchen Sinne) ist gleich der Anzahl der verschiedenen mikroskopischen Zustände, durch welche der makroskopische Zustand realisiert gedacht werden kann. Die Entropie des makroskopischen Zustandes und damit das statistische und thermodynamische Verhalten des Systems wird dann durch den BOLZMANSschen Satz bestimmt.

#### § 1. Die Zellen.

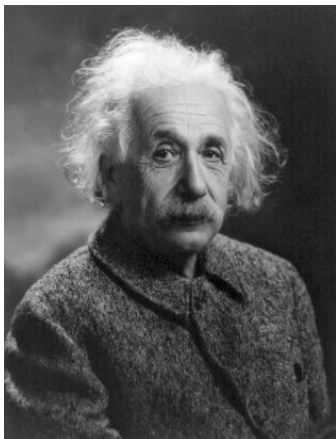
Das Phasenvolumen, welches zu einem gewissen Bereich der Koordinaten  $x, y, z$  und zugehörigen Momente  $p_x, p_y, p_z$  eines einatomigen Moleküls gehört, wird durch das Integral

$$\Phi = \int dx dy dz dp_x dp_y dp_z \quad (1)$$

ausgedrückt. Ist  $V$  das dem Molekül zur Verfügung stehende Volumen, so ist das Phasenvolumen aller Zustände, deren Energie  $E = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$  kleiner ist als ein bestimmter Wert  $E$ , gegeben durch

$$\Phi = V \cdot \frac{4}{3} \pi (2mE)^{\frac{3}{2}}. \quad (1a)$$

<sup>1</sup> Erscheint nächstens in der »Zeitschr. für Physik«.



EINSTEIN: Quantentheorie des einatomigen idealen Gases

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### § 3. Thermodynamisches Gleichgewicht.

Beim thermodynamischen Gleichgewicht ist  $S$  ein Maximum, wobei außer (3) den Nebenbedingungen zu genügen ist, daß die Gesamtzahl  $n$  der Atome sowie deren Gesamtenergie  $\bar{E}$  gegebene Werte besitzen. Diese Bedingungen drücken sich offenbar in den beiden Gleichungen aus<sup>1</sup>

$$n = \sum_r r p_r' \quad (6)$$

$$\bar{E} = \sum_r E^r r p_r' \quad (7)$$

wobei  $E^r$  die Energie eines Moleküls bedeutet, welches zur  $r$ ten Phasenzelle gehört. Aus (1a) folgt man leicht, daß

$$\left. \begin{aligned} E^r &= cs^{\frac{2}{3}} \\ c &= (2m)^{-1} h^2 \left( \frac{4}{3} \pi V \right)^{-\frac{2}{3}} \end{aligned} \right\} \quad (8)$$

Durch Ausführung der Variation nach den  $p_r'$  als Variablen findet man, daß bei passender Wahl der Konstanten  $\beta', A$  und  $B$

$$\left. \begin{aligned} p_r' &= \beta' e^{-\alpha^r r} \\ \alpha^r &= A + B s^{\frac{2}{3}} \end{aligned} \right\} \quad (9)$$

sein muß. Gemäß (3) muß hierbei sein

$$\beta' = 1 - e^{-\alpha^r}. \quad (10)$$

Hieraus ergibt sich zunächst für die mittlere Zahl der Moleküle pro Zelle

$$n^r = \sum_r r p_r' = \beta' \sum_r r e^{-\alpha^r r} = -\beta' \frac{d}{d\alpha^r} (\sum_r e^{-\alpha^r r}) = -\beta' \frac{d}{d\alpha^r} \left( \frac{1}{1 - e^{-\alpha^r}} \right) = \frac{1}{e^{\alpha^r} - 1}. \quad (11)$$

# Einstein , 1925

So what happens when we increase the density of the substance,  $n/V$ , at constant temperature (for example by isothermal compression)?

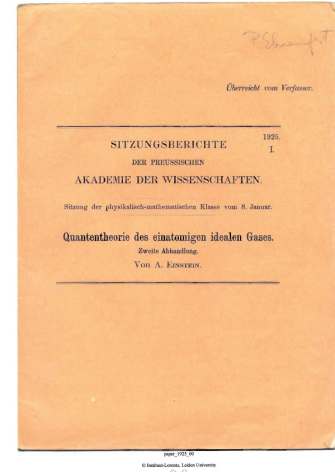
I suggest that in this case, as you increase the overall density, an ever increasing number of molecules drop into the ground state (i.e. the first quantum state, state of no kinetic energy)<sup>[2]</sup>, while the other molecules will distribute according to the parameter  $\lambda=1$ .

My prediction is the emergence of something very similar to what occurs when vapor is isothermally compressed beyond the saturation volume. A separation will take place; one part will “condense”, the rest will remain as a “saturated ideal gas” ( $A=0 \lambda=1$ )

.....  
Hence we acquire this theorem: According to the derived state equation of ideal gases, for any given temperature there exists a maximum density of agitated molecules.

London, F. (1938a), *Nature* **141**, 643.

London, F. (1938b), *Phys. Rev.* **54**, 947.

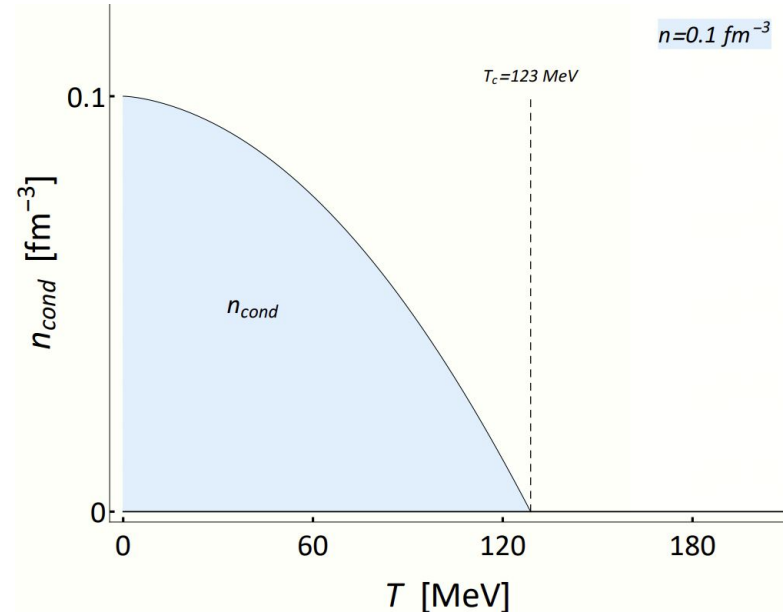
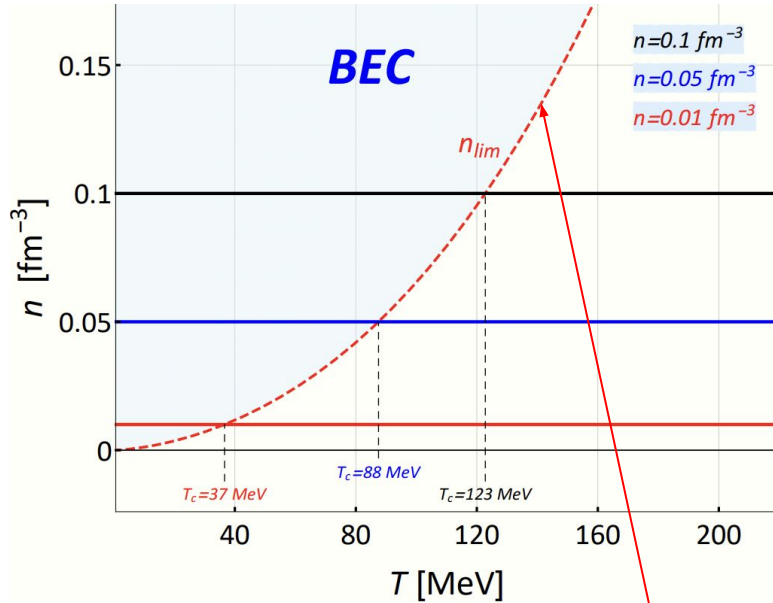


Quantentheorie des einatomigen idealen Gases.  
Zweite Abhandlung.  
Von A. EINSTEIN.

In einer neulich in diesen Berichten (XXII 1924, S. 261) erschienenen Abhandlung wurde unter Anwendung einer von Hrn. D. Bose zur Ableitung der Planckschen Strahlungsformel erdachten Methode eine Theorie der Entartung idealer Gase angegeben. Das Interesse dieser Theorie liegt darin, daß sie auf die Hypothese einer weitgehenden formalen Verwandtschaft zwischen Strahlung und Gas gegründet ist. Nach dieser Theorie weicht das entartete Gas von dem Gas der mechanischen Statistik in analoger Weise ab wie die Strahlung gemäß dem Planckschen Gesetze von der Strahlung gemäß dem Wignerschen Gesetze. Wenn die Bosesche Ableitung der Planckschen Strahlungsformel ernst genommen wird, so wird man auch an dieser Theorie des idealen Gases nicht vorbeigehen dürfen; denn wenn es gerechtfertigt ist, die Strahlung als Quantengas aufzufassen, so muß die Analogie zwischen Quantengas und Molekülgas eine vollständige sein. Im folgenden sollen die früheren Überlegungen durch einige neue ergänzt werden, die mir das Interesse an dem Gegenstande zu steigern scheinen. Der Bequemlichkeit halber schreibe ich das Folgende formal als Fortsetzung der zitierten Abhandlung.

- P. Ehrenfest, Phasenumwandlungen im ueblichen und erweiterten Sinn, classifiziert nach dem entsprechenden Singularitaeten des thermodynamischen Potentiales, Communications from the Physical Laboratory of the University of Leiden, Supplement No. 75b (1933).
- Gregg Jaeger, The Ehrenfest Classification of Phase Transitions: Introduction and Evolution, Arch. Hist. Exact Sci. 53, 51 (1998).
- M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, and E.A. Cornell, Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor, Science 269, 198 (1995).
- C.C. Bradley, C.A. Sackett, J.J. Tollett, and R.G. Hulet, Evidence of Bose-Einstein Condensation in an Atomic Gas with Attractive Interactions, Phys. Rev. Lett. 75, 1687 (1995).

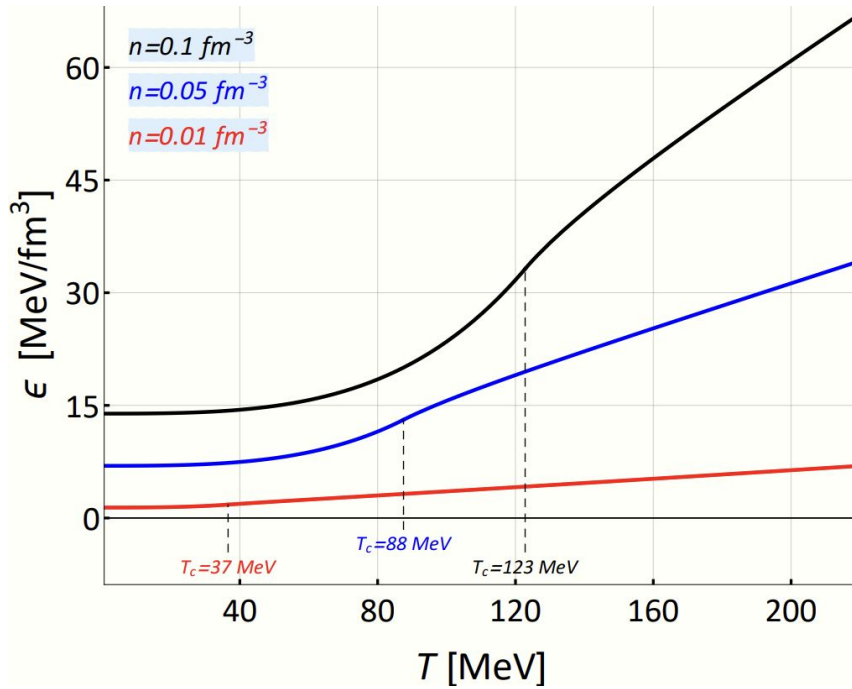
# 1-component ideal gas



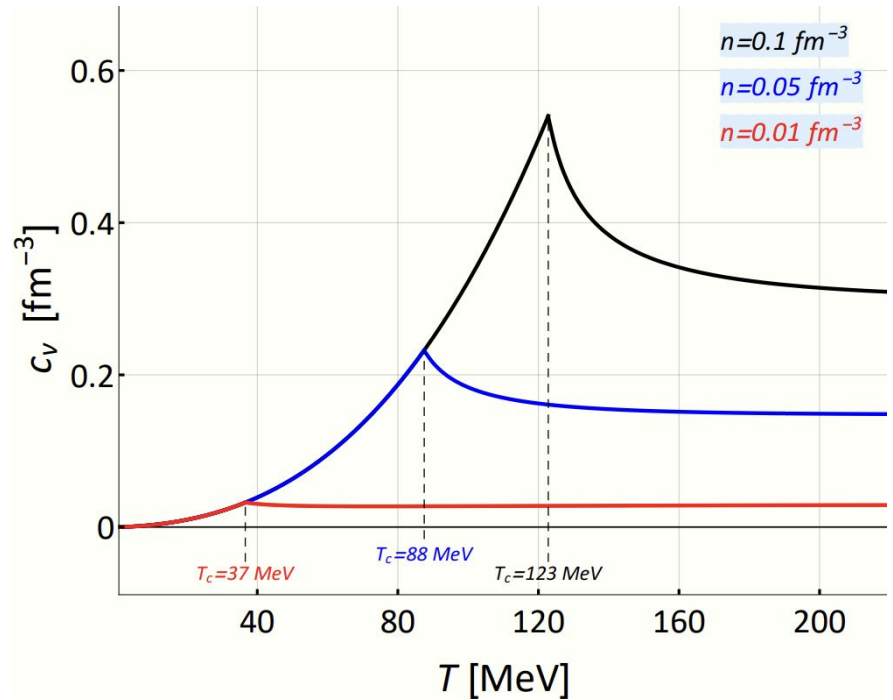
$$n_{lim} = g \int_0^\infty \frac{d^3k}{(2\pi)^3} \frac{1}{\exp\left[\frac{\sqrt{k^2 + m^2} - m}{T}\right] - 1}$$

Dependence of the density of the meson condensate (shaded area) on temperature.

# 1-component ideal gas

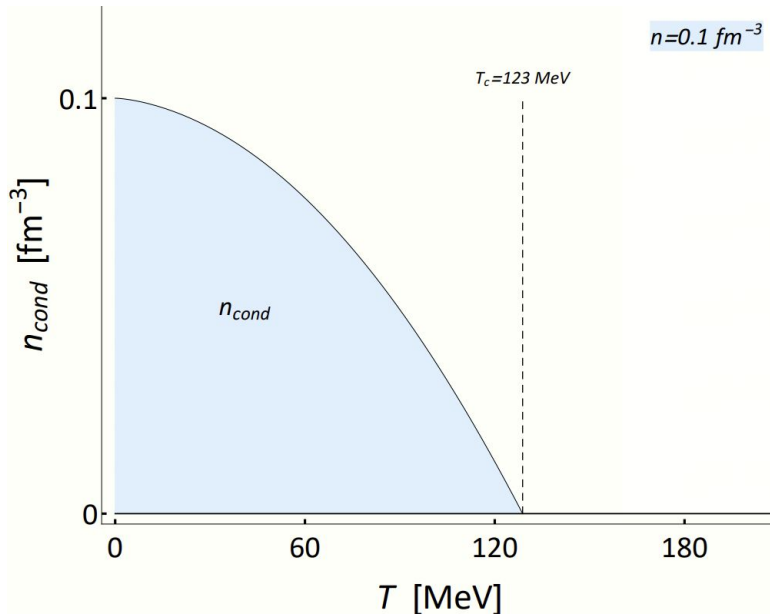


Energy density vs. temperature for the meson system.

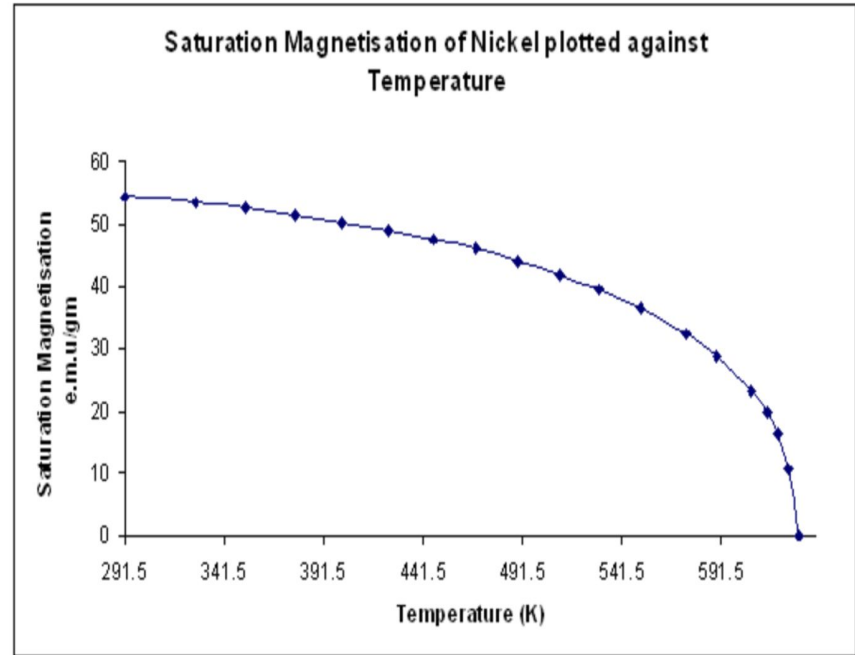


Heat capacity vs. temperature for the meson system.

# 1-component ideal gas



Dependence of the density of the meson condensate (shaded area) on temperature.



Variation of saturation magnetization with temperature for Nickel. (Data from Weiss and Forrer, 1926)

- P. Kerson Huang, *Statistical Mechanics*, John Wiley and Sons, 1987; (Section 12.3, p. 294).  
"the real conservation law deals with the conserved quantity that is the number of particles minus the number of antiparticles. That is why any study of the Bose-Einstein condensation in the relativistic Bose gas must take antiparticles into account."
- H. E. Haber and H. A. Weldon, *Thermodynamics of an Ultrarelativistic Ideal Bose Gas*, *Phys. Rev. Lett.* **46**, 1497 (1981).

J. Kapusta. Bose-Einstein condensation, spontaneous symmetry breaking, and gauge theories. *Phys. Rev. D* **24**, 426 (1981) [DOI: <https://doi.org/10.1103/PhysRevD.24.426>].

H.E. Haber, H.A. Weldon. Finite-temperature symmetry breaking as Bose-Einstein condensation. *Phys. Rev. D* **25**, 502 (1982) [DOI: <https://doi.org/10.1103/PhysRevD.25.502>].

J. Bernstein, S. Dodelson. Relativistic Bose gas. *Phys. Rev. Lett.* **66**, 683 (1991) [DOI: <https://doi.org/10.1103/PhysRevLett.66.683>].

## Condensation of interacting scalar bosons

$$\mathcal{L}(x) = \frac{1}{2} \left[ \partial_\mu \hat{\phi}(x) \partial^\mu \hat{\phi}(x) - m^2 \hat{\phi}^2(x) \right] + \mathcal{L}_{\text{int}}[\hat{\phi}^2(x)]$$

We adopt the famous Bogolyubov's decomposition of the field operator into two contributions [1]

$$\hat{\phi}(\mathbf{r}) = \phi_{\text{cond}} + \hat{\psi}(\mathbf{r}), \quad \text{where} \quad \langle \hat{\psi}(\mathbf{r}) \rangle = 0.$$

### Mean scalar density

$$\langle \hat{\phi}^+ \hat{\phi} \rangle = \langle |\Phi_0|^2 + \Phi_0^* \hat{\psi} + \Phi_0 \hat{\psi}^+ + \hat{\psi}^+ \hat{\psi} \rangle = |\Phi_0|^2 + \langle \hat{\psi}^+ \hat{\psi} \rangle$$

- [1] N. Bogolubov, On the theory of superfluidity, *Sov. J. Phys.* **11**, 23 (1947).  
M.M. Bogolyubov, *Lekciji z kvantovoyi statystyky*, Kyiv, 1947 (Ukrainian).  
N.N. Bogoliubov, *Lectures on Quantum Statistics*, Gordon and Breach, New York, 1967.



$$\phi^+ \phi \equiv \hat{\sigma},$$

$$\langle \phi^+ \phi \rangle \equiv \sigma.$$

$$\mathcal{L}_{\text{int}}(\hat{\sigma}) = \frac{a}{2} \hat{\sigma}^2 - \frac{b}{3} \hat{\sigma}^3$$

$$\langle \phi^+ \phi \rangle \rightarrow n, \quad \langle \phi^+ \phi \rangle^2 \rightarrow n^2, \quad \langle \phi^+ \phi \rangle^3 \rightarrow n^3$$

$$U(\sigma) = -a\sigma + b\sigma^2 \rightarrow U(n) = -An + Bn^2$$

Thermodynamic mean-field model:  
description of an interacting  
2-component relativistic system



# Thermodynamic mean-field model<sup>[1-3]</sup>

Starting  
point

Free energy density:  $\phi = F/V$

$$\phi(n_1, n_2, T) = \phi_1^{(0)}(n_1, T) + \phi_2^{(0)}(n_2, T) + \phi_{\text{int}}(n, T),$$

[1] D. Anchishkin and V. Vovchenko, J. Phys. G 42, 105102 (2015).

[2] D. Anchishkin, I. Mishustin, and H. Stoecker, J. Phys. G 46, 035002 (2019).

[3] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Universe 2023, 9, 411 (2023).

# Thermodynamic mean-field model

Free energy density:

$$\phi(n_1, n_2, T) = \phi_1^{(0)}(n_1, T) + \phi_2^{(0)}(n_2, T) + \phi_{\text{int}}(n, T),$$



Starting point

Self-consistent set of equations

$$n = \int \frac{d^3k}{(2\pi)^3} [f_{\text{BE}}(E(k, n), \mu_I) + f_{\text{BE}}(E(k, n), -\mu_I)] ,$$

$$n_I = \int \frac{d^3k}{(2\pi)^3} [f_{\text{BE}}(E(k, n), \mu_I) - f_{\text{BE}}(E(k, n), -\mu_I)] .$$

$$E(k, n) = \sqrt{m^2 + \mathbf{k}^2} + U(n)$$

$$f_{\text{BE}}(E, \mu) = \left\{ \exp\left[\frac{E - \mu}{T}\right] - 1 \right\}^{-1}$$

# Thermodynamic mean-field model

Free energy density:

$$\phi(n_1, n_2, T) = \phi_1^{(0)}(n_1, T) + \phi_2^{(0)}(n_2, T) + \phi_{\text{int}}(n, T),$$



Starting  
point

Self-consistent set  
of equations

$$n = \int \frac{d^3k}{(2\pi)^3} [f_{\text{BE}}(E(k, n), \mu_I) + f_{\text{BE}}(E(k, n), -\mu_I)] ,$$

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$$E(k, n) = \sqrt{m^2 + \mathbf{k}^2} + U(n)$$

Fixed

$$f_{\text{BE}}(E, \mu) = \left\{ \exp\left[\frac{E - \mu}{T}\right] - 1 \right\}^{-1}$$

# Thermodynamic mean-field model

Starting point

Free energy density:

$$\phi(n_1, n_2, T) = \phi_1^{(0)}(n_1, T) + \phi_2^{(0)}(n_2, T) + \phi_{\text{int}}(n, T),$$



Self-consistent set of equations

$$n = \int \frac{d^3k}{(2\pi)^3} [f_{\text{BE}}(E(k, n), \mu_I) + f_{\text{BE}}(E(k, n), -\mu_I)],$$

$$n_I = \int \frac{d^3k}{(2\pi)^3} [f_{\text{BE}}(E(k, n), \mu_I) - f_{\text{BE}}(E(k, n), -\mu_I)].$$

$$E(k, n) = \sqrt{m^2 + \mathbf{k}^2} + U(n)$$

Fixed

$$f_{\text{BE}}(E, \mu) = \left\{ \exp\left[\frac{E - \mu}{T}\right] - 1 \right\}^{-1}$$



Parameterization of the mean-field

$$m = 140 \text{ MeV}$$

$$U(n) = -An + Bn^2, \quad U(n) + m = 0 \quad n_1 = \sqrt{\frac{m}{B}}(\kappa - \sqrt{\kappa^2 - 1}), \quad n_2 = \sqrt{\frac{m}{B}}(\kappa + \sqrt{\kappa^2 - 1}),$$

$$\kappa \equiv \frac{A}{2\sqrt{mB}}.$$

# Thermodynamic mean-field model

Starting point

Free energy density:

$$\phi(n_1, n_2, T) = \phi_1^{(0)}(n_1, T) + \phi_2^{(0)}(n_2, T) + \phi_{\text{int}}(n, T),$$



Self-consistent set of equations

$$n = \int \frac{d^3k}{(2\pi)^3} [f_{\text{BE}}(E(k, n), \mu_I) + f_{\text{BE}}(E(k, n), -\mu_I)],$$

$$n_I = \int \frac{d^3k}{(2\pi)^3} [f_{\text{BE}}(E(k, n), \mu_I) - f_{\text{BE}}(E(k, n), -\mu_I)].$$

$$E(k, n) = \sqrt{m^2 + \mathbf{k}^2} + U(n)$$

Fixed  $f_{\text{BE}}(E, \mu) = \left\{ \exp\left[\frac{E - \mu}{T}\right] - 1 \right\}^{-1}$



Parameterization of the mean-field

$$m = 140 \text{ MeV}$$

$$U(n) = -An + Bn^2, \quad U(n) + m = 0 \quad n_1 = \sqrt{\frac{m}{B}}(\kappa - \sqrt{\kappa^2 - 1}), \quad n_2 = \sqrt{\frac{m}{B}}(\kappa + \sqrt{\kappa^2 - 1}),$$

$$\kappa \equiv \frac{A}{2\sqrt{mB}}.$$

Interplay between A and B  
Parameter B - fixed

# Thermodynamic mean-field model

Starting point

Free energy density:

$$\phi(n_1, n_2, T) = \phi_1^{(0)}(n_1, T) + \phi_2^{(0)}(n_2, T) + \phi_{\text{int}}(n, T),$$



Self-consistent set of equations

$$n = \int \frac{d^3k}{(2\pi)^3} [f_{\text{BE}}(E(k, n), \mu_I) + f_{\text{BE}}(E(k, n), -\mu_I)],$$

$$n_I = \int \frac{d^3k}{(2\pi)^3} [f_{\text{BE}}(E(k, n), \mu_I) - f_{\text{BE}}(E(k, n), -\mu_I)].$$

$$E(k, n) = \sqrt{m^2 + \mathbf{k}^2} + U(n)$$

Fixed  $f_{\text{BE}}(E, \mu) = \left\{ \exp\left[\frac{E - \mu}{T}\right] - 1 \right\}^{-1}$



Parameterization of the mean-field

$$m = 140 \text{ MeV}$$

$$U(n) = -An + Bn^2, \quad U(n) + m = 0 \quad n_1 = \sqrt{\frac{m}{B}}(\kappa - \sqrt{\kappa^2 - 1}), \quad n_2 = \sqrt{\frac{m}{B}}(\kappa + \sqrt{\kappa^2 - 1}),$$

$$\kappa \equiv \frac{A}{2\sqrt{mB}}$$

Interplay between A and B  
Parameter B - fixed

$\kappa < 1$  - weak attraction regime

$\kappa = 1$  - critical attraction regime

$\kappa > 1$  - strong(over-critical) attraction regime



## Particle-antiparticle system with conservation of isospin (charge)

Parametrization of  $U(n)$ :

$$\langle \phi^+ \phi \rangle \rightarrow n, \quad \langle \phi^+ \phi \rangle^2 \rightarrow n^2, \quad \langle \phi^+ \phi \rangle^3 \rightarrow n^3$$

$$U(\langle \phi^+ \phi \rangle) = -a \langle \phi^+ \phi \rangle + b \langle \phi^+ \phi \rangle^2 \rightarrow U(n) = -An + Bn^2$$

$$P_{\text{ex}}(n) = -\frac{1}{2}An^2 + \frac{2}{3}Bn^3$$

Parameterization of the attraction coefficient:

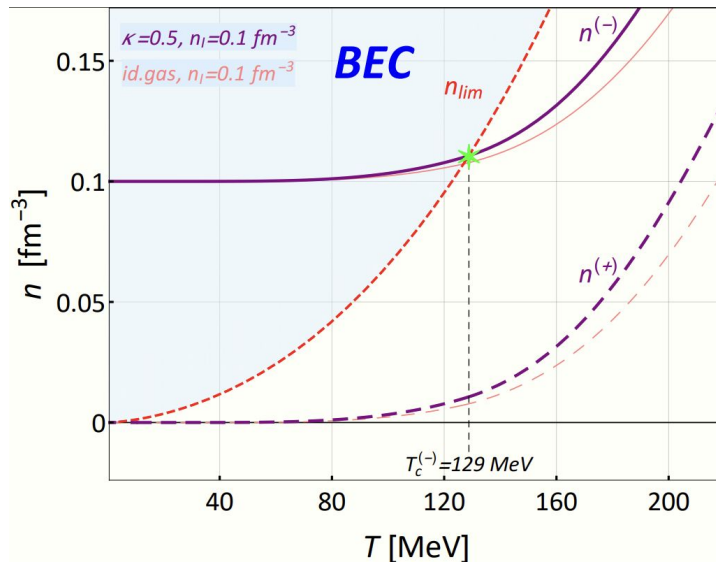
$$U(n) + m = 0$$

$$n_1 = \sqrt{\frac{m}{B}} \left( \kappa - \sqrt{\kappa^2 - 1} \right), \quad n_2 = \sqrt{\frac{m}{B}} \left( \kappa + \sqrt{\kappa^2 - 1} \right)$$

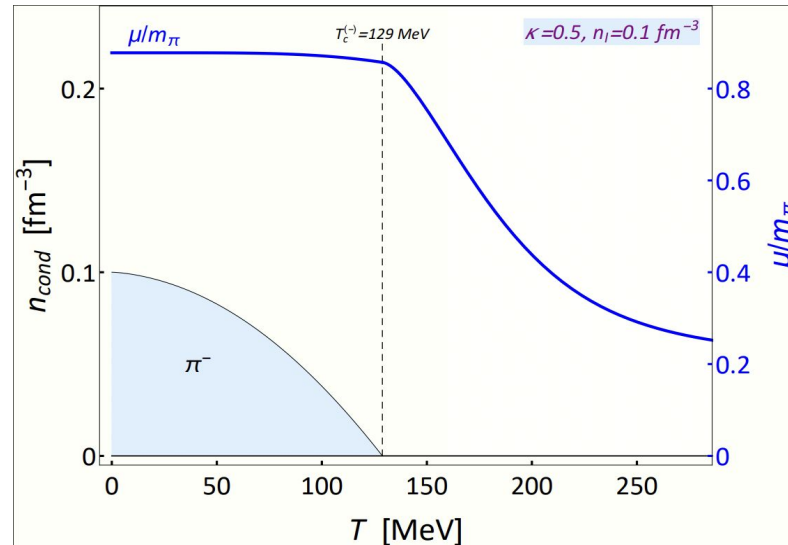
$$\kappa \equiv \frac{A}{2\sqrt{mB}}, \quad A_c = 2\sqrt{mB}, \quad A = \kappa A_c$$

# Type 1 phase transition (2nd order ph. tr.)

Weak attraction regime  $\kappa = 0.5$



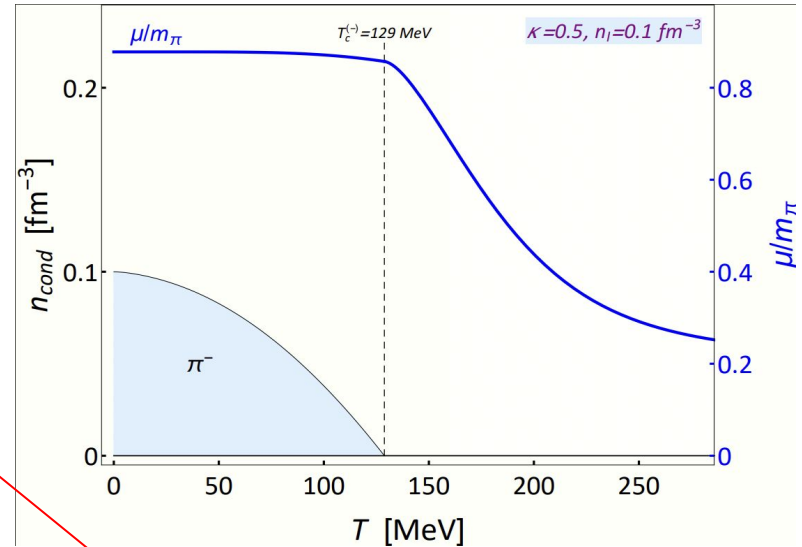
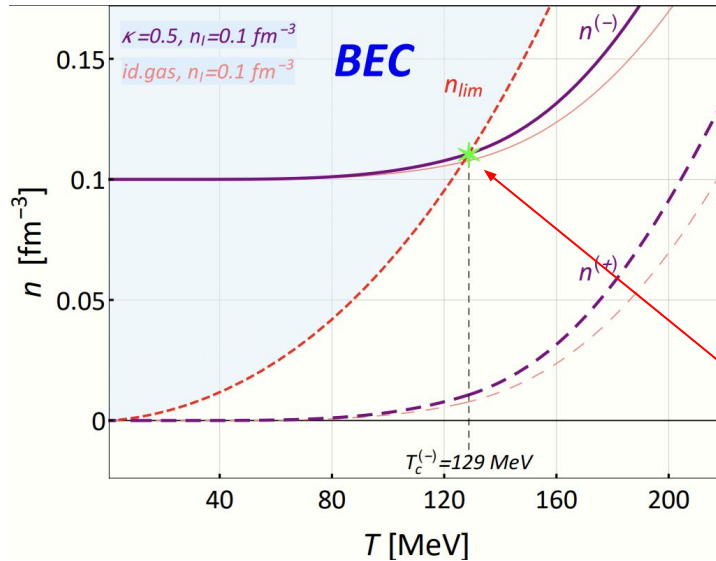
**Figure 1.** The particle-number densities  $n^{(+)}$ ,  $n^{(-)}$  versus temperature for the interacting  $\pi^+\pi^-$  pion gas in the mean-field model. The total isospin density is kept constant,  $n_l = 0.1 \text{ fm}^{-3}$ , and the attraction parameter is  $\kappa = 0.5$ .



Dependence of the density of the  $\pi^-$  meson condensate (shaded area) and the chemical potential on temperature.

# Type 1 phase transition (2nd order ph. tr.)

Weak attraction regime  $\kappa = 0.5$



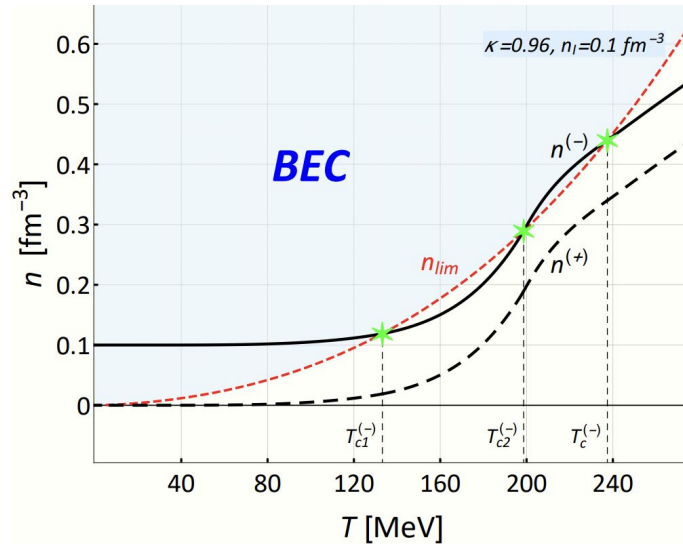
Phase transition of the 2nd order

**Figure 1.** The particle-number densities  $n^{(+)}$ ,  $n^{(-)}$  versus temperature for the interacting  $\pi^+\pi^-$  pion gas in the mean-field model. The total isospin density is kept constant,  $n_I = 0.1$  fm<sup>-3</sup>, and the attraction parameter is  $\kappa = 0.5$ .

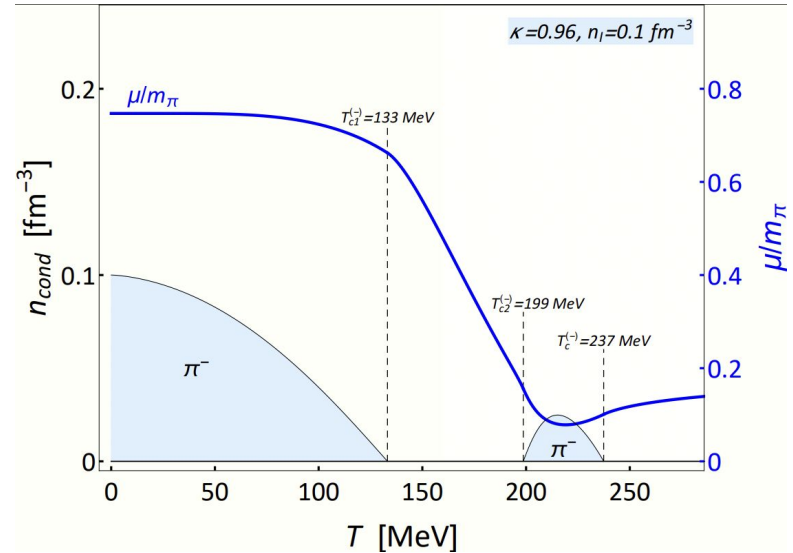
What will happen if we increase the attraction parameter  $\kappa$  in such a system?

# Type 2 phase transition (multiple ph.tr., 2nd order ph. tr.)

Weak attraction regime  $\kappa = 0.96$



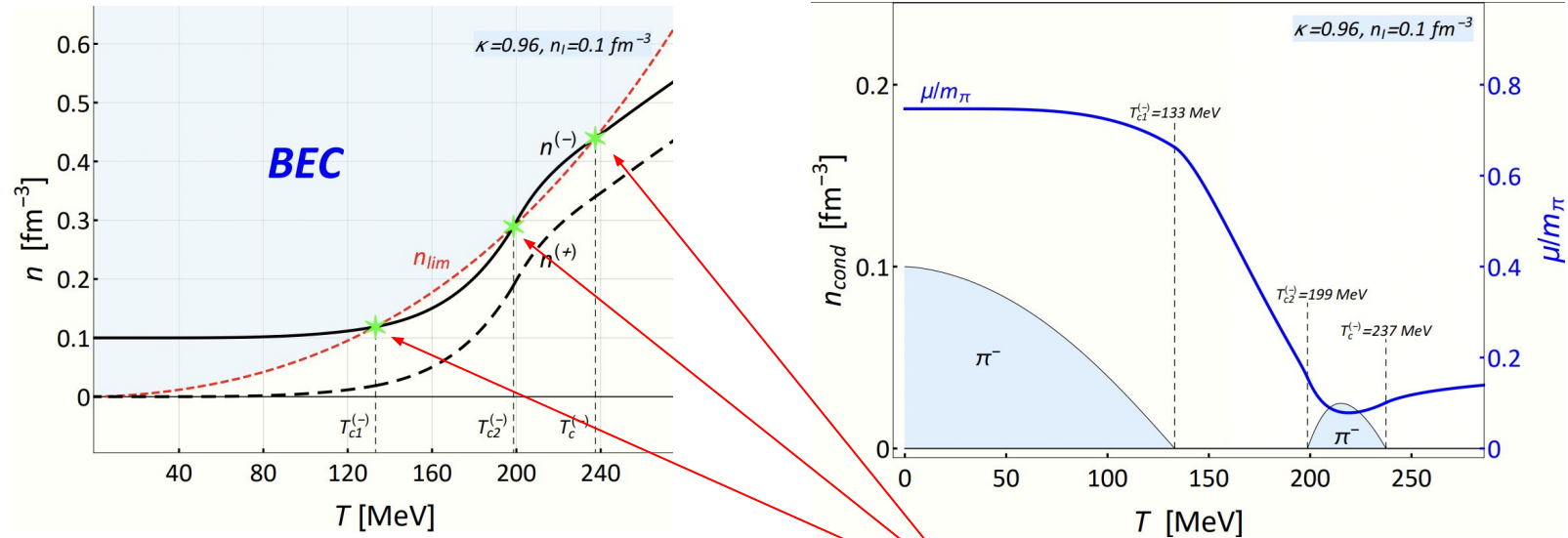
**Figure 2.** The particle-number densities  $n^{(+)}$ ,  $n^{(-)}$  versus temperature for the interacting  $\pi^+\pi^-$  pion gas in the mean-field model. The total isospin density is kept constant,  $n_1 = 0.1 \text{ fm}^{-3}$ , and the attraction parameter is  $\kappa = 0.96$ .



Dependence of the density of the  $\pi^-$  meson condensate (shaded areas) and the chemical potential on temperature.

# Type 2 phase transition (multiple ph.tr., 2nd order ph. tr.)

Weak attraction regime  $\kappa = 0.96$

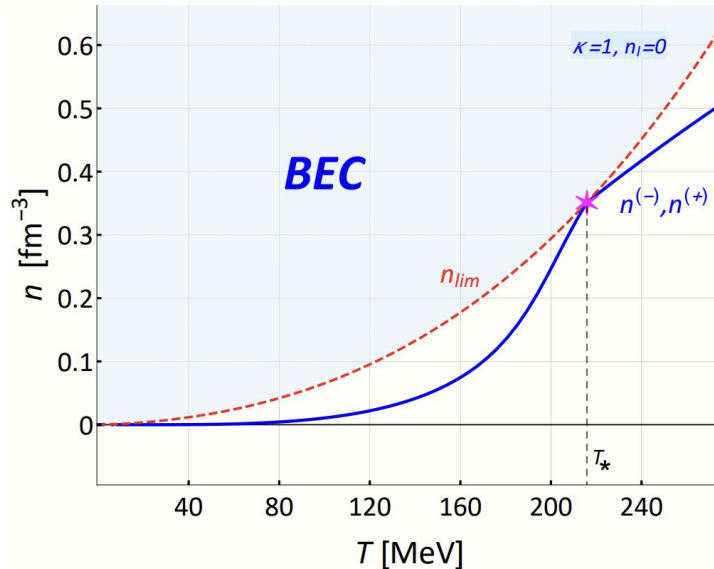


**Figure 2.** The particle-number densities  $n^{(+)}$ ,  $n^{(-)}$  versus temperature for the interacting  $\pi^+\pi^-$  pion gas in the mean-field model. The total isospin density is kept constant,  $n_l = 0.1$  fm<sup>-3</sup>, and the attraction parameter is  $\kappa = 0.96$ .

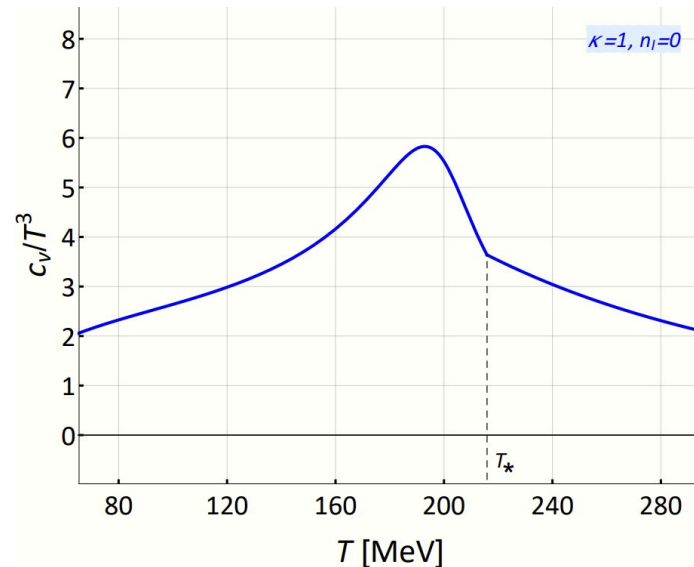
Phase transition of the 2nd order

# Type 3 phase transition (meta ph.tr., 2nd order ph. tr.)

Critical attraction regime  $\kappa = 1$



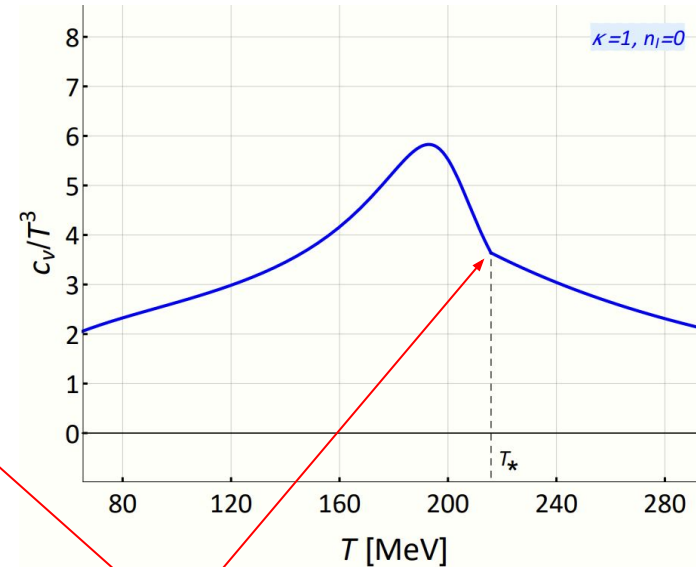
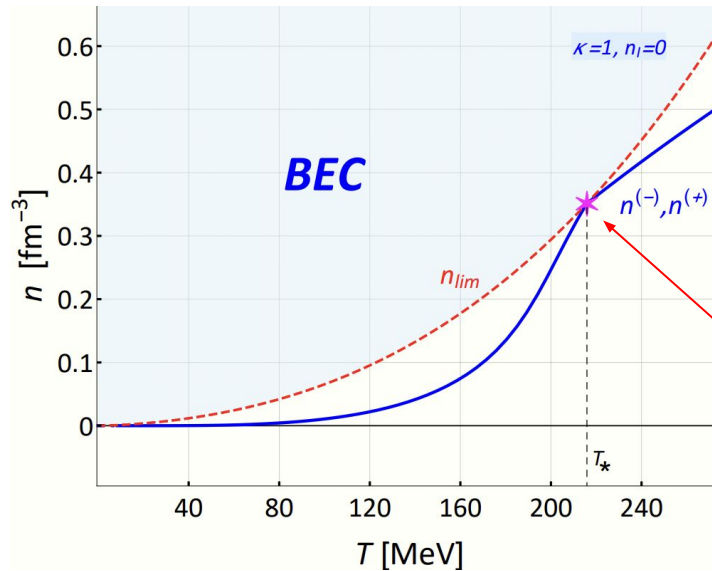
**Figure 3.** The particle-number densities  $n^{(+)}$ ,  $n^{(-)}$  versus temperature for the interacting  $\pi^+\pi^-$  pion gas in the mean-field model. The total isospin density is kept constant,  $n_l = 0$ , and the attraction parameter is  $\kappa = 1$ .



Heat capacity versus temperature for the interacting  $\pi^+\pi^-$  pion gas in the mean-field model. The total isospin density is kept constant,  $n_l = 0$ . The temperature  $T^*$  indicates the meta phase transition of the second order.

# Type 3 phase transition (meta ph. tr., 2nd order ph. tr.)

Critical attraction regime  $\kappa = 1$



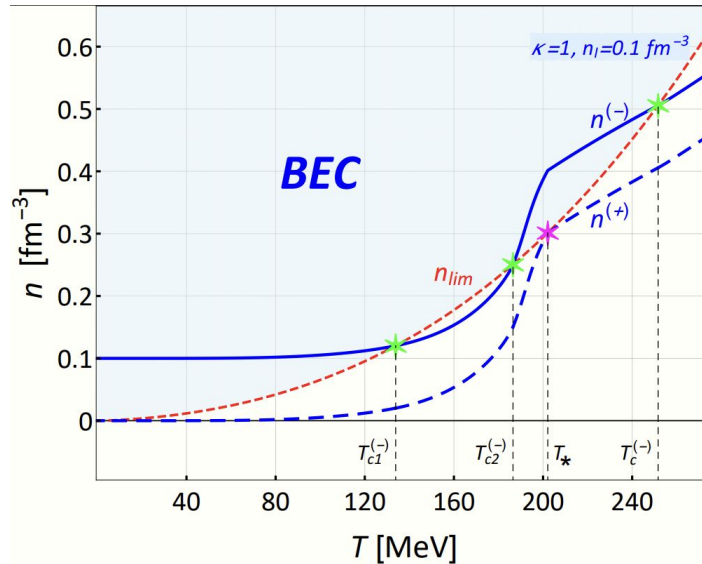
**Figure 3.** The particle-number densities  $n^{(+)}$ ,  $n^{(-)}$  versus temperature for the interacting  $\pi^+\pi^-$  pion gas in the mean-field model. The total isospin density is kept constant,  $n_l = 0$ , and the attraction parameter is  $\kappa = 1$ .

Phase transition of the 2nd order

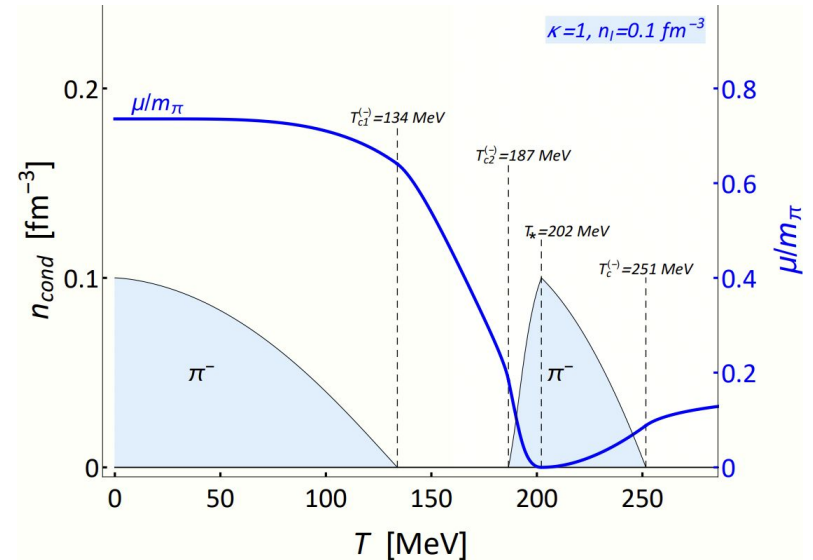


# Type 3 phase transition (meta ph. tr., 2nd order ph. tr.)

Critical attraction regime  $\kappa = 1$



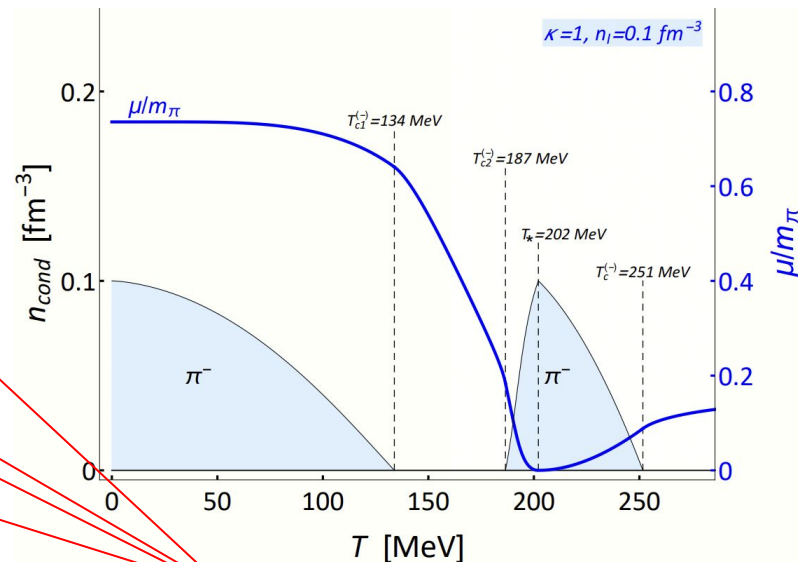
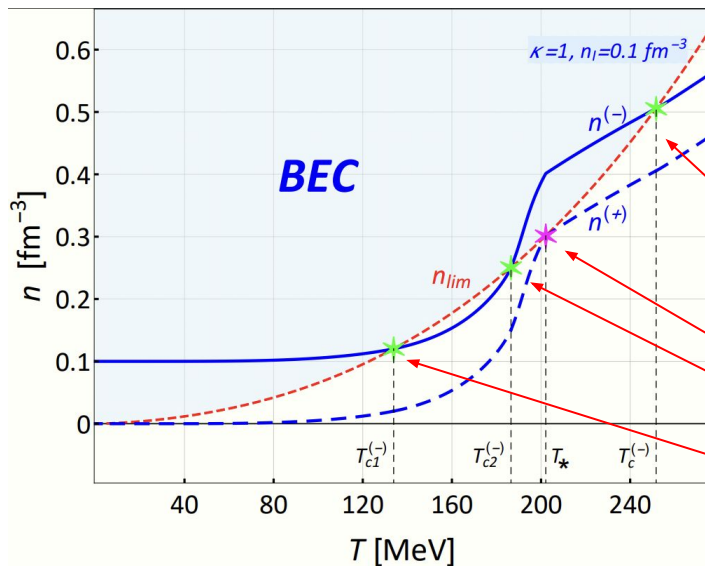
**Figure 4.** The particle-number densities  $n^{(+)}$ ,  $n^{(-)}$  versus temperature for the interacting  $\pi^{+}\text{-}\pi^{-}$  pion gas in the mean-field model. The total isospin density is kept constant,  $n_l = 0.1 \text{ fm}^{-3}$ , and the attraction parameter is  $\kappa = 1$ .



Dependence of the density of the  $\pi^{-}$  meson condensate (shaded areas) and the chemical potential on temperature.

# Type 3 phase transition (meta ph. tr., 2nd order ph. tr.)

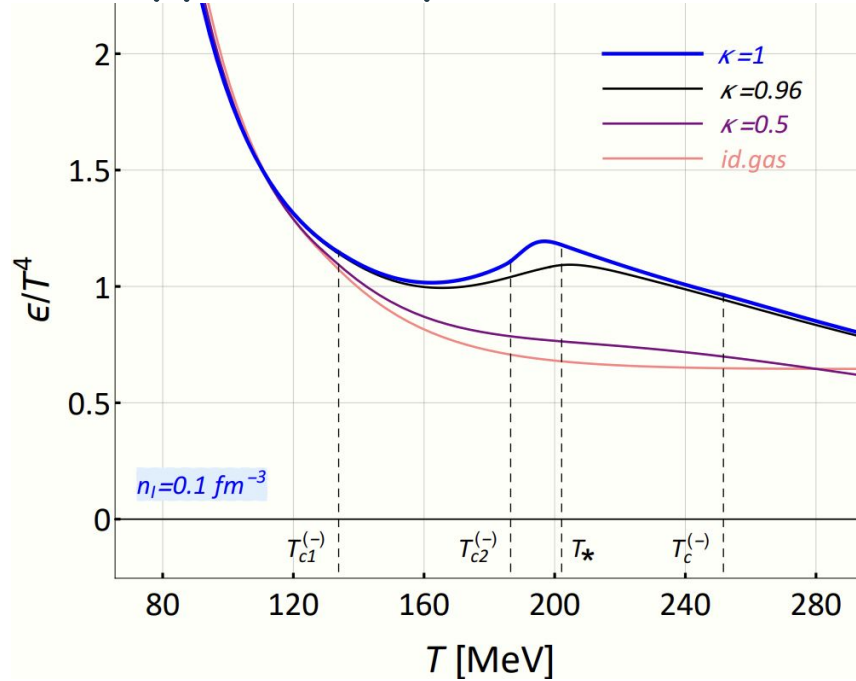
Critical attraction regime  $\kappa = 1$



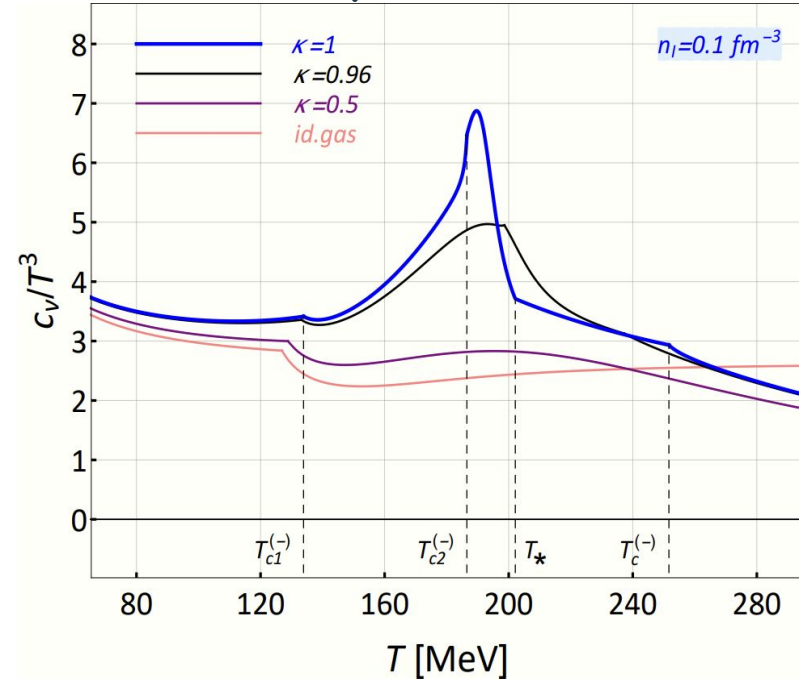
**Figure 4.** The particle-number densities  $n^{(+)}$ ,  $n^{(-)}$  versus temperature for the interacting  $\pi^{+}\text{-}\pi^{-}$  pion gas in the mean-field model. The total isospin density is kept constant,  $n_I = 0.1 \text{ fm}^{-3}$ , and the attraction parameter is  $\kappa = 1$ .

Phase transition of the 2nd order

# Type 1,2,3 phase transition (2nd order ph. tr.)<sup>[3]</sup>



**Figure 5.** Energy density versus temperature for the interacting  $\pi^+\pi^-$  pion gas in the mean-field model. The total isospin density is kept constant,  $n_l = 0.1 \text{ fm}^{-3}$ .



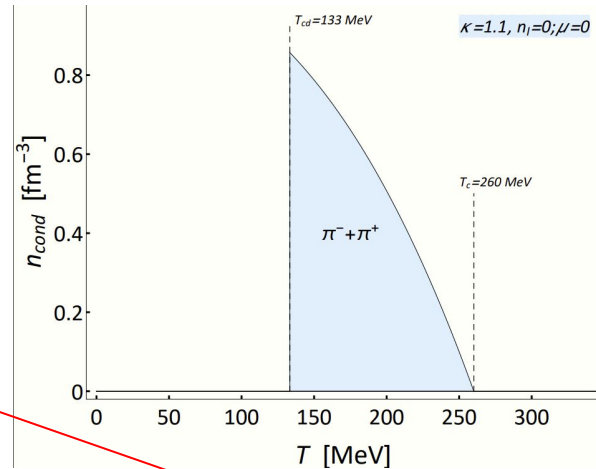
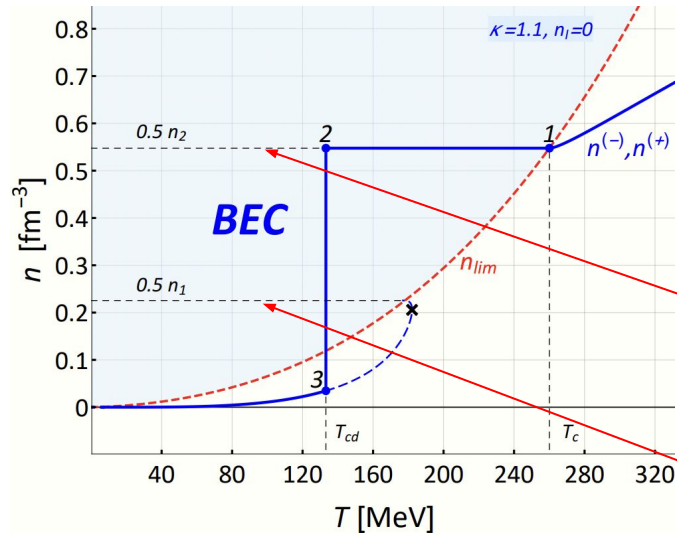
**Figure 6.** Heat capacity versus temperature for the interacting  $\pi^+\pi^-$  pion gas in the mean-field model. The total isospin density is kept constant,  $n_l = 0.1 \text{ fm}^{-3}$ .

What if  $\kappa$  is in over-critical regime?

$$\kappa > 1$$

# Type 4 phase transition (1st order ph. tr.)<sup>[3,4]</sup>

Over-critical attraction regime  $\kappa = 1.1$



**Figure 7.** The particle-number densities  $n^{(+)}$ ,  $n^{(-)}$  versus temperature for the interacting  $\pi^+\pi^-$  pion gas,  $n_l = 0$ . The attraction parameter is  $\kappa = 1.1$ .

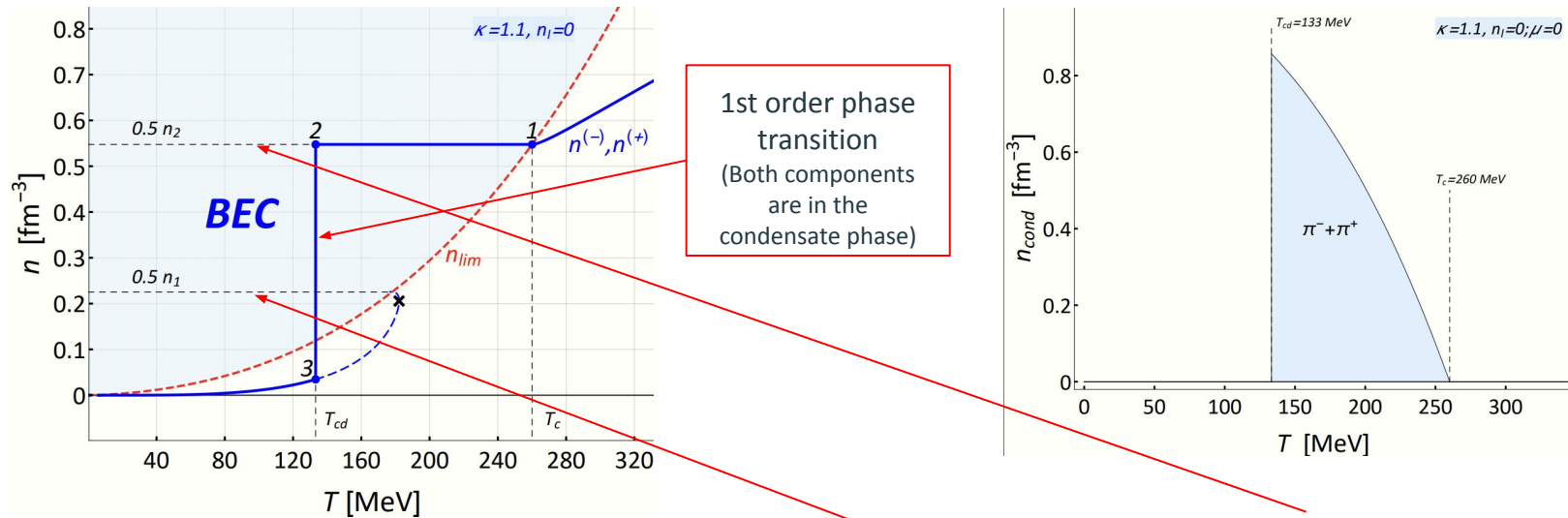
$$n_1 = \sqrt{\frac{m}{B}} (\kappa - \sqrt{\kappa^2 - 1}), \quad n_2 = \sqrt{\frac{m}{B}} (\kappa + \sqrt{\kappa^2 - 1}),$$

[3] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Universe 2023, 9, 411 (2023).

[4] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Ukr. J. Phys. 2024. Vol. 69, No. 1 (2024).

# Type 4 phase transition (1st order ph. tr.)<sup>[3,4]</sup>

Over-critical attraction regime  $\kappa = 1.1$



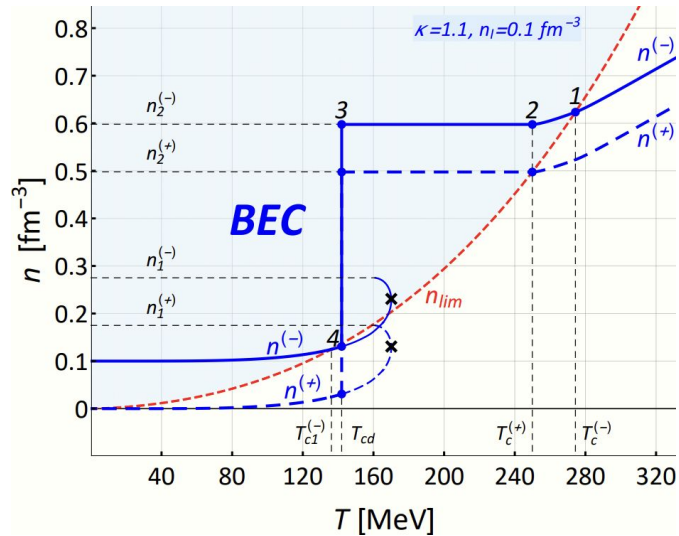
**Figure 7.** The particle-number densities  $n^{(+)}$ ,  $n^{(-)}$  versus temperature for the interacting  $\pi^+\pi^-$  pion gas,  $n_l = 0$ . The attraction parameter is  $\kappa = 1.1$ .

$$n_1 = \sqrt{\frac{m}{B}} (\kappa - \sqrt{\kappa^2 - 1}), \quad n_2 = \sqrt{\frac{m}{B}} (\kappa + \sqrt{\kappa^2 - 1}),$$

[3] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Universe 2023, 9, 411 (2023).  
 [4] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Ukr. J. Phys. 2024. Vol. 69, No. 1 (2024).

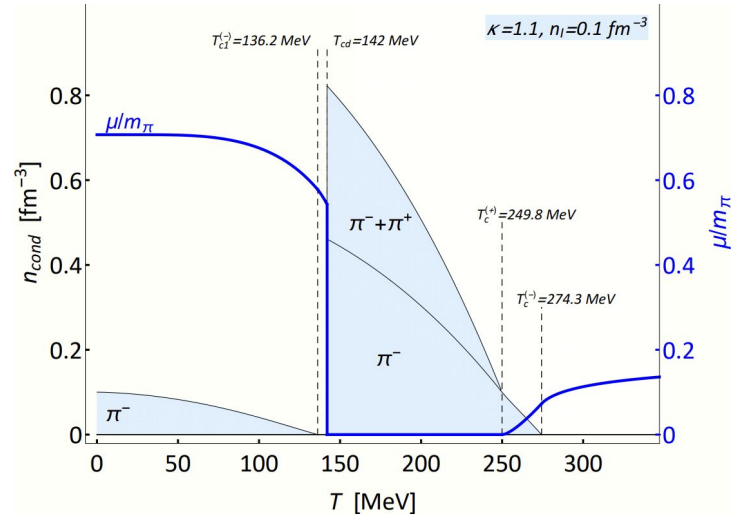
# Type 4 phase transition (1st order ph. tr.)<sup>[3,4]</sup>

Over-critical attraction regime  $\kappa = 1.1$



$$n_1^{(-)} - n_1^{(+)} = n_l,$$

$$n_2^{(-)} - n_2^{(+)} = n_l.$$



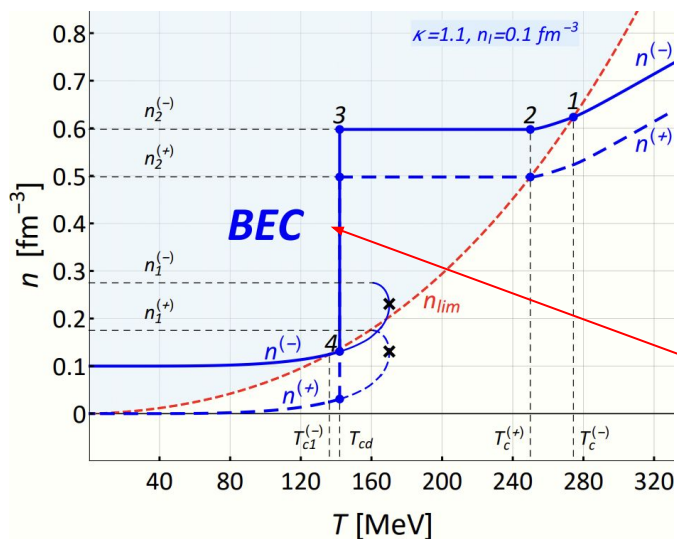
**Figure 8.** The particle-number densities  $n^{(+)}$ ,  $n^{(-)}$  versus temperature for the interacting  $\pi^+ - \pi^-$  pion gas,  $n_l = 0.1 \text{ fm}^{-3}$ . The attraction parameter is  $\kappa = 1.1$ .

[3] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Universe 2023, 9, 411 (2023).

[4] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Ukr. J. Phys. 2024. Vol. 69, No. 1 (2024).

# Type 4 phase transition (1st order ph. tr.)<sup>[3,4]</sup>

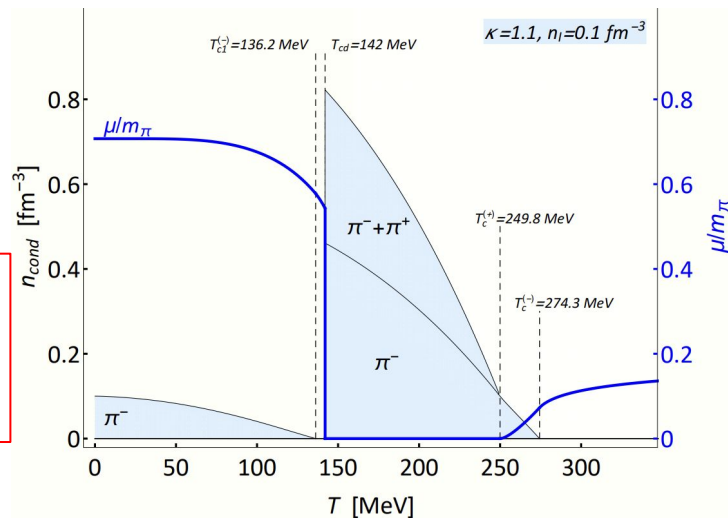
Over-critical attraction regime  $\kappa = 1.1$



$$n_1^{(-)} - n_1^{(+)} = n_l,$$

$$n_2^{(-)} - n_2^{(+)} = n_l.$$

1st order phase transition  
(Both components are in the condensate phase)



**Figure 8.** The particle-number densities  $n^{(+)}$ ,  $n^{(-)}$  versus temperature for the interacting  $\pi^+ \pi^-$  pion gas,  $n_l = 0.1 \text{ fm}^{-3}$ . The attraction parameter is  $\kappa = 1.1$ .

[3] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Universe 2023, 9, 411 (2023).

[4] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Ukr. J. Phys. 2024. Vol. 69, No. 1 (2024).



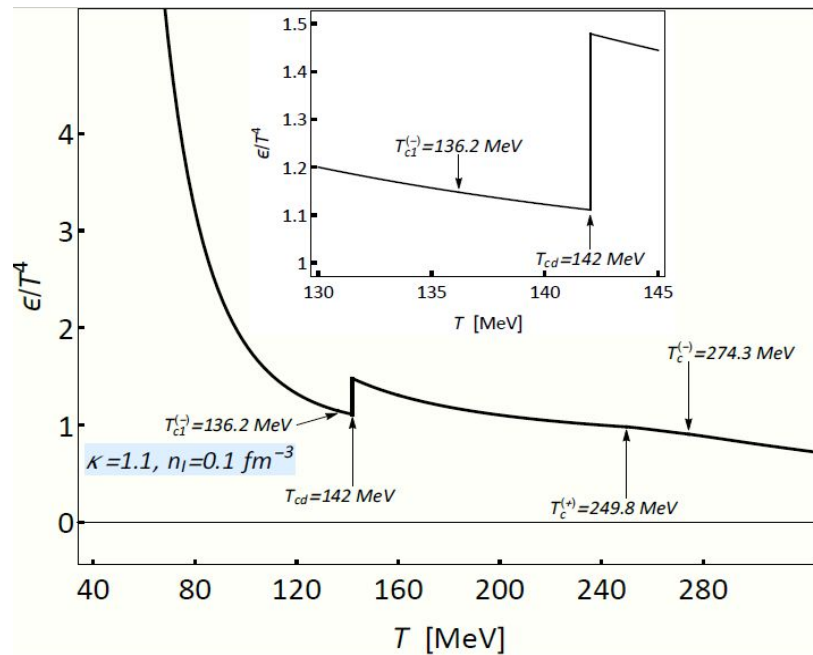
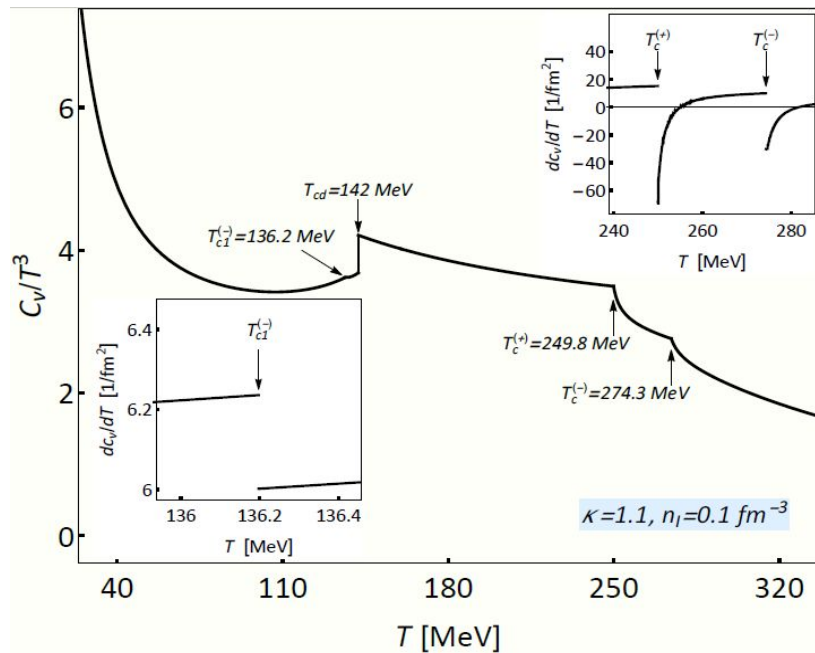
# Conclusions

- ❑ The thermodynamic mean-field model was used to describe the 2-component system of interacting mesons (bosons) at high temperatures in the presence of condensate.
- ❑ In the case of the conservation of isospin, 4 types of phase transitions are defined in a relativistic bosonic system with repulsion and attraction: **type 1** - the one-time transition of the dominant component (2nd order), **type 2** - the multiple phase transition of the dominant component (2nd order), **type 3** - the meta phase transition of the second component (2nd order), **type 4** - the transition of both components into the condensate phase (1st order).
- ❑ When both components are in the condensate phase, the thermal particles have no additional charge, that is, the isospin (charge) density of the thermal particles is zero. Additional charge accumulates only in condensate particles.

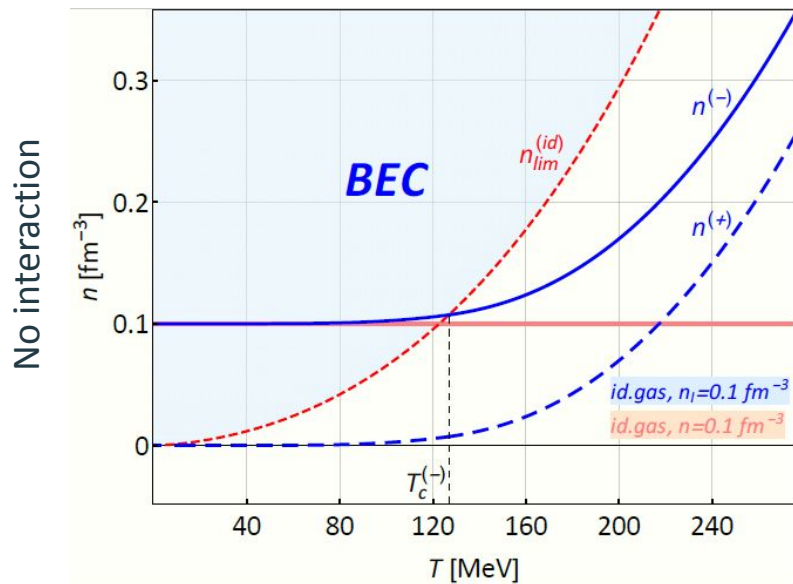
Thank you for attention!

# Type 4 phase transition (1st order ph. tr.)<sup>[3,4]</sup>

## Proof



# Ideal 2-component relativistic gas



**Figure 1.** The particle-number densities  $n^{(+)}$ ,  $n^{(-)}$  versus temperature for the ideal  $\pi^+\pi^-$  pion gas. The total isospin density is kept constant,  $n_I = 0.1 \text{ fm}^{-3}$ .

$$n_I = \int \frac{d^3k}{(2\pi)^3} [f_{\text{BE}}(\omega_k, \mu_I) - f_{\text{BE}}(\omega_k, -\mu_I)]$$

$$\omega_k = \sqrt{m^2 + \mathbf{k}^2}$$

Both components cannot be in the condensate phase at the same time:

$$\begin{aligned} m - \mu_I &= 0, \\ m + \mu_I &= 0 \end{aligned}$$



$$m = 0$$

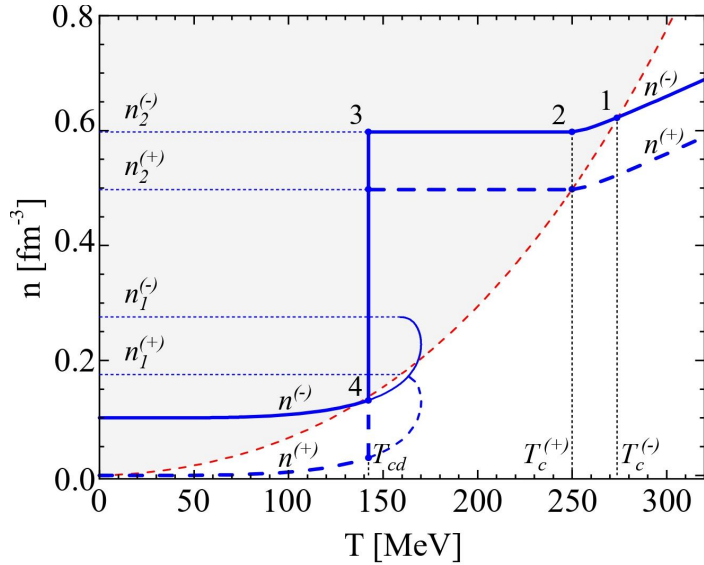
$$\mu_I = 0$$

Condition for the condensate formation:

$$\mu_I = m$$

In the interval  $0 < T < T_c$

## In the condensate phase the chemical potential is not a free parameter



GCE: canonical variables are  $(T, \mu_I)$

CE: canonical variables are  $(T, n_I)$

The thermodynamic mean-field model

$$m + U(n) - \mu_I = 0$$

$$m + U(n) + \mu_I = 0$$



$$\mu_I = 0$$

$$m + U(n) = 0$$

The scalar field model

$$M(\sigma) - \mu_I = 0$$

$$M(\sigma) + \mu_I = 0$$



$$\mu_I = 0$$

$$M(\sigma) = 0$$